EGM703 – Advanced Active and Passive Remote Sensing

Week 3, Supplement 1: Complex Numbers

Polynomials

- Any polynomial with degree n should have n roots
  - In other words, if $f(x)$ is a polynomial with degree n, there are n numbers (not necessarily unique!) where $f = 0$.
  - Example: $f(x) = x^2 - 1$ has roots $+1$, -1:

$$\begin{aligned} f(x) = x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

- Simple, right?

Complex numbers

- What about $f(x) = x^2 + 1$?
- As before:

$$f(x) = x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \pm\sqrt{-1} = i$$

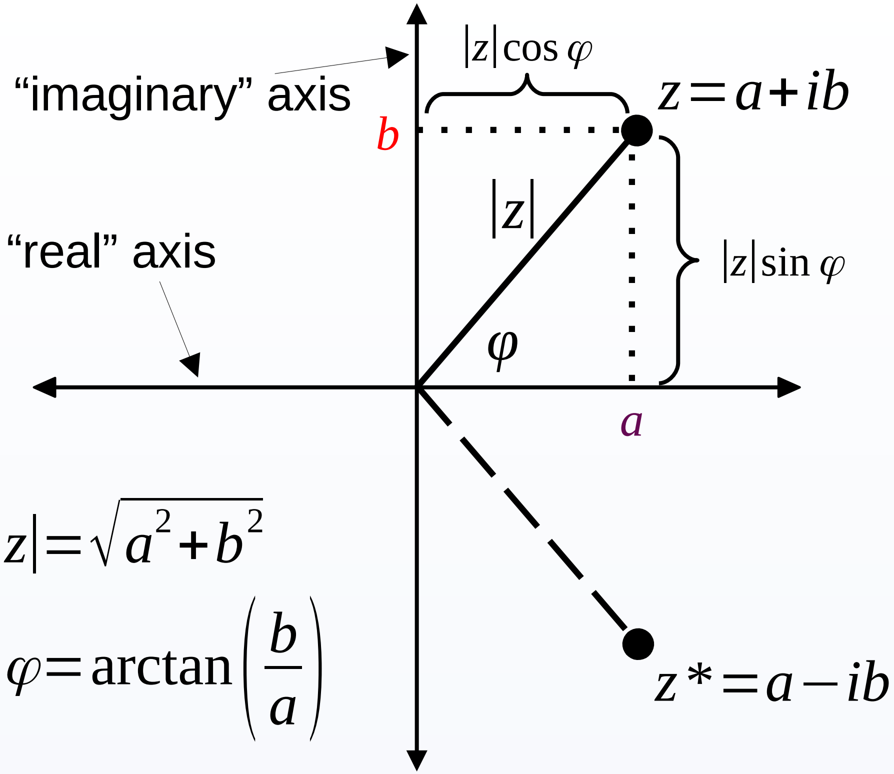
- The roots z of $f(x) = x^2+1$ are complex (i.e., $z \in \mathbb{C}$)
- A complex number z can be written:

$$z = \boxed{a} + i\boxed{b} \qquad a, b \in \mathbb{R}$$

Real part
Imaginary part

Complex numbers as vectors

- We can also think about z as a vector with components a, b
  - Length (modulus or magnitude), $|z|$
  - Angle (argument), φ
- The complex conjugate of z, z^*, is the reflection of z across the "real" axis
  - In other words, multiply b by -1



$$|z| = \sqrt{a^2 + b^2}$$

$$\varphi = \arctan\left(\frac{b}{a}\right)$$

Complex notation

- We can write z in other ways, as well:
  - Component notation:

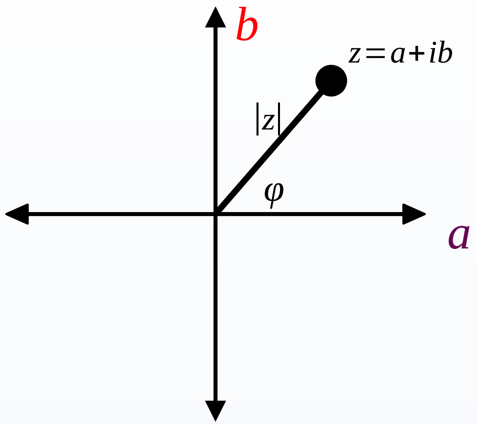
$$z = a + ib$$

  - Polar notation:

$$z = |z|(\cos\varphi + i\sin\varphi)$$

  - Euler notation:

$$z = |z| e^{i\varphi}$$



Complex arithmetic: addition

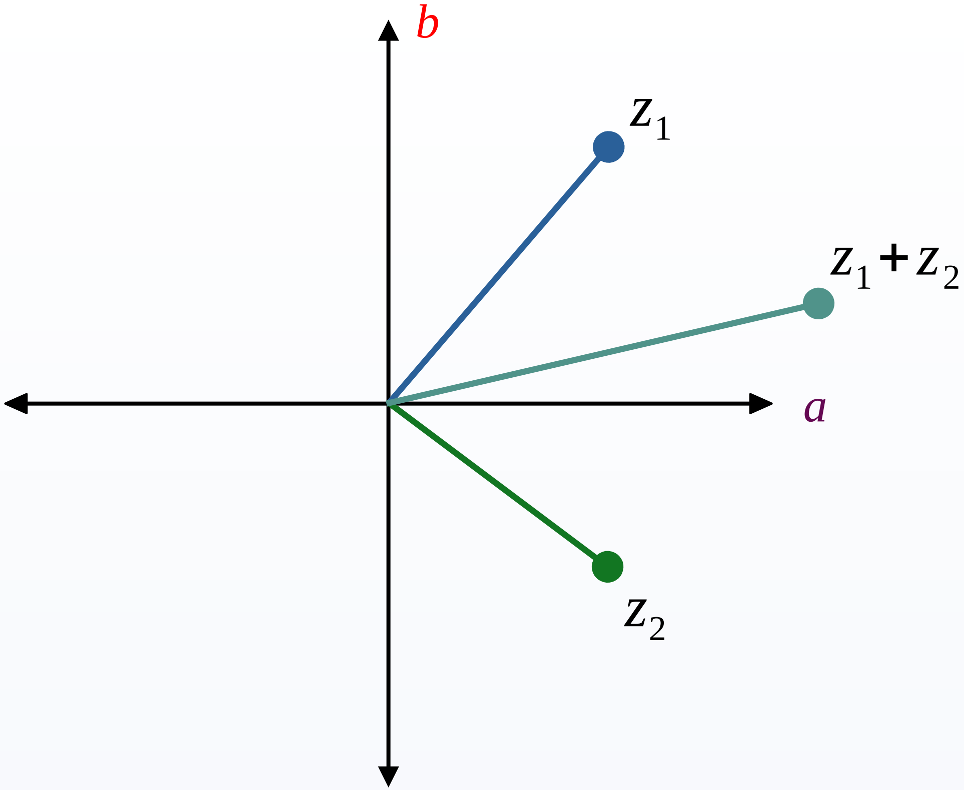
- This is just vector addition:

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$$

$$= (a_1 + a_2) + i(b_1 + b_2)$$

$$|z_1 + z_2| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$$

$$arg(z_1 + z_2) = \arctan\left(\frac{b_1 + b_2}{a_1 + a_2}\right)$$



Complex arithmetic: multiplication

- Multiplication is slightly more complicated:

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$$

that's just -1!

$$= a_1 a_2 + a_1 i b_2 + i b_1 a_2 + \boxed{i^2} b_1 b_2$$

$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

- Alternatively:

$$z_1 z_2 = |z_1| e^{i\varphi_1} |z_2| e^{i\varphi_2}$$

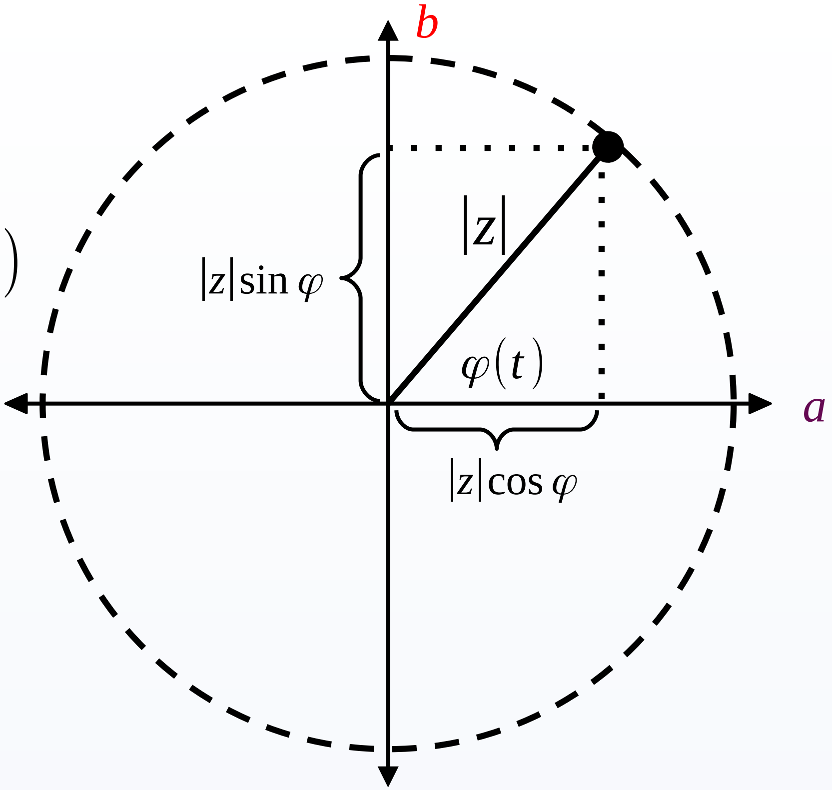
$$= |z_1||z_2| e^{i(\varphi_1 + \varphi_2)}$$

Complex numbers and oscillating signals

- An oscillating signal can be represented:

$$u(t) = |z| e^{i\varphi(t)} = |z|(\cos(\varphi(t)) + i\sin(\varphi(t)))$$

- Relevant examples:
  - Electromagnetic waves (generally)
  - Radar signals



Summary

- Complex numbers:
  - Are just another way of doing arithmetic
  - Simplify a great deal of complicated math
  - Can be used to represent oscillating signals (like waves!)
  - Have some other uses that we'll see moving forward