Slide 1 – Title Slide

Hello and welcome to Week 3, Supplement 1 of EGM703: Complex Numbers. In this lesson, we'll discuss the mathematical concept of complex numbers, and explain how these can be used in remote sensing, and specifically in radar remote sensing.

Slide 2 – Polynomials

Before we dive right in, we're going to start with polynomials. As you will hopefully recall, any polynomial with degree n should have n roots. In other words, if f(x) is a polynomial with degree n, there are n numbers, not necessarily unique, where f is equal to zero. For example, if f(x) is equal to x^2 minus 1, f has two roots, at +1 and -1. f(1) is equal to 0, as is f(-1). To check this, we can set f(x) equal to zero, and solve for x: it's hopefully clear that x = ±1 is the solution here. Okay, so far so good, yes?

Slide 3 – Complex numbers

Okay, so what happens if we have $f(x) = x^2 + 1$? Let's try to solve this like we did before: first, we set f equal to zero, but now we have $x^2 = -1$. We know that ordinarily, we can't take the square root of a negative number. But, we also know that f has to have two roots – f is a polynomial, and we can't just selectively decide that polynomials are only polynomials in certain cases. So, we have to move beyond the "conventional" math we might be used to – we can define the square root of -1 to be a number, i, and then the roots of f are ±i. We say that the roots, z, of this polynomial are complex – you may also see it written this way – this notation just means that z is part of, or "in", the set of complex numbers, C. We can also write a complex number z as follows: z = a + ib, where a and b are "real" numbers (in other words, the regular old floating-point numbers that we're used to). We say that a is the "real" part of z, and b is the "imaginary" part. Don't get too caught up on the names here – despite the name, the 'imaginary' part of z is every bit as important and real as the 'real' part.

Slide 4 – Complex numbers as vectors

We can also think about z as being a vector that has components a and b. The horizontal axis is also known as the "real" axis, because the value on this axis corresponds to the "real" part of z, while the vertical axis is also known as the "imaginary" axis, because the value on this axis corresponds to the "imaginary" part of z. The vector z then has a length, also known as the modulus or magnitude, denoted as |z| - this is equal to the square root of the sum of the squares of the real and imaginary parts of z. The vector z makes an angle with the real axis, also known as the argument of z, denoted here with the greek letter phi. We can calculate phi as the arctangent of the imaginary part of z divided by the real part of z. From this representation, we can also see that the real part of z is equal to the magnitude of z multiplied by the cosine of the angle of z, while the imaginary party is equal to the magnitude multiplied by the sine of the angle of z. We can also define the complex conjugate of z, denoted z* - this is the reflection of z across the "real" axis – in other words, we multiply b by -1.

Slide 5 – Complex notation

In addition to thinking of z as a vector, we can also write z in other ways, as well. We've seen the complex notation already, where z is equal to a plus i times b; we can also write z in polar notation – here, we're just replacing a with the magnitude multiplied by the cosine of the angle, and replacing b with the magnitude by the sine of the angle, and re-arranging by factoring out the magnitude. It turns out that this value here, cos phi + i times sin phi, is equal to euler's number, e, raised to the power of i times phi. Using this identity, we can write z using Euler notation, where z is equal to the magnitude of z times e to the i times phi.

Slide 6 – Complex arithmetic: addition

Okay, so what if we want to add two complex numbers together? Well, because we can represent complex numbers as vectors, this is just vector addition. So, if we have a number, z1, and another number, z2, then z1 + z2 is just the sum of these two vectors. If we write this out using the components, we can group the real parts together like this, and the imaginary parts together like this – the new real part is just the sum of the two real parts, and the new imaginary part is just the sum of the two imaginary parts. The magnitude of this vector is the square root of the sum of the squares its real and imaginary parts, and the angle, or argument, is the arctangent of the imaginary part divided by the real part.

Slide 7 – Complex arithmetic: multiplication

Okay, so addition doesn't look so bad. Multiplication, though, gets very complicated very quickly, if we continue using this notation. If we multiple z1 by z2, we have to multiply each of these different parts out – so we end up with a1 times a2, plus a1 times ib2, plus ib1 times a2, plus i^2 times b1 times b2. Remember that i^2 is just -1, so we can re-organize this and write it out as follows: the real part of z1 times z2 is a1 times a2 minus b1 times b2, while the imaginary part is a1 times b2 plus a2 times b1. On the other hand, if we were to write this using Euler notation, we would have the following: this is just the magnitude of z1 times the magnitude of z2, times e to i times the sum of the angles of z1 and z2.

Slide 8 – Complex numbers and oscillating signals

Now we'll see why it is that we're learning about complex numbers in a remote sensing class. Let's think about a signal that is oscillating as a function of time – we'll call it u(t). If we look at this as a vector with some magnitude and an angle which varies in time, we can see that this will trace out a circle around the origin: at one point in time the vector will point this direction, at another point in time it will point in this direction, and so on. This is just the same as the complex numbers we've been looking at: the real component of this signal is given by the magnitude of z multiplied by the cosine of the angle, while the imaginary component of this signal is given by the magnitude of z multiplied by the sine of the angle. So, we can represent this oscillating signal, u(t), the same way that we've represented complex numbers. It turns out, we're already pretty familiar with oscillating signals: as we've covered in a few different lessons at this point, electromagnetic radiation is a wave that oscillates in time. And, as we'll see in the lessons this week, radar signals, which are a specific kind of electromagnetic radiation, can also be represented in this way.

Slide 9 – Summary

In this lesson, we've discussed how complex numbers aren't scary, they're really just another way of doing arithmetic. They simplify a lot of really complicated math, and we can use them to represent oscillating signals, like waves. In addition, in next week's lecture, we'll see some other ways that we can use complex numbers in remote sensing applications. That's all for this lesson – I hope you found it interesting, and if you have any questions, please don't hesitate to e-mail me or post in the discussion forum on blackboard. Bye!